Method of excess fractions with application to absolute distance metrology: theoretical analysis

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The method of excess fractions (EF) is well established to resolve the fringe order ambiguity generated in interferometric detection. Despite this background, multiwavelength interferometric absolute long distance measurements have only been reported with varying degrees of success. In this paper we present a theoretical model that can predict the unambiguous measurement range in EF based on the selected measurement wavelengths and phase noise. It is shown that beat wavelength solutions are a subset of this theoretical model. The performance of EF, for a given phase noise, is shown to be equivalent to beat techniques but offers many alternative sets of measurement wavelengths and therefore EF offer significantly greater flexibility in experimental design.

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1. Introduction

The precise measurement of distance is of major importance in science and technology. In 1892, Michelson *et al.* carried out the first measurement of the length of the Pt-Ir bar, which was the international prototype of the meter by using the red cadmium wavelength in an interferometer [1]. Since then, the challenge faced by optical metrologists is to provide solutions for absolute long-range distance measurements to nanometer precision [2–5]. Examples of long range metrology include the measurement of gauge blocks up to 500 mm in standards laboratories [6,7] and in astronomy where the distance between the arms in an interferometric telescope array can be tens of meters [8–10].

In a Michelson interferometer, the optical path difference (OPD) of a single fringe corresponds to a displacement of $\lambda/2$. For example, an optical fringe in the *C*-band represents a displacement of a few hundred nanometers. Subfringe resolution can be achieved routinely using phase stepping or Fourier transform techniques [11,12] while, more recently, sinusoidal modulation methods have also been introduced [13,14]. The typical resolution attained is between 1/100th and 1/1000th of a fringe. However, the subfringe data repeats every fringe, and therefore, to obtain measurements beyond $\lambda/2$ becomes a rather difficult task. The major challenge is then to determine the unknown integer fringe order. One approach has been to incorporate the necessary electronics or algorithms to count the interference orders sequentially [15]. Nevertheless, fringe counting is time consuming, making it unsuitable for dynamic applications and also sensitive to environmental disturbances and system drifts.

Multiwavelength interferometry (MWI) provides a way to overcome the $\lambda/2$ range limit [16–18]. In MWI, an independent phase measurement is made at each wavelength, and these data are combined to give an unambiguous measure over an extended range which is defined by the synthetic beat wavelength [18]. The largest synthetic wavelength of the MWI system defines the unambiguous measurement range (UMR) and is inversely proportional to the smallest wavelength separation. However, the optical signals have to be demultiplexed to individual

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detectors, which for small wavelength separations represent a practical limitation. For instance, when using C-band wavelengths, a minimum wavelength separation of ~2 pm is required in order to achieve a measurement range of 1 m.

For a two-wavelength interferometer and in the presence of phase noise, the ratio between the beat wavelength and the smallest measurement wavelength must be limited in order to calculate the correct fringe order within the beat [19]. The addition of further measurement wavelengths may result in an extended measurement range. The wavelength selection is not trivial as it affects both the measurement range and the reliability of the fringe order calculation. Several approaches employ a geometric series of synthetic wavelengths (beats) [20–23] in order to cover a large dynamic range between the longest beat wavelength and the measurement wavelengths. Towers *et al.* [20] maximized the scaling factor between the terms of the geometric series of beat wavelengths in order to minimize the number of measurement wavelengths for a given measurement range and phase noise, referred to as generalized optimum multifrequency (GOMF) interferometry. However, GOMF defines a unique set of wavelengths, and the measurement range is equal to the largest beat wavelength and hence may be impractical particularly for long range applications where the wavelength separation required becomes too small to facilitate demultiplexing and in some cases the precise measurement wavelengths may be unavailable. More recently, the authors have reported a hybrid geometric series approach [24] that employs the extended range method of de Groot [25] to overcome the requirement for two very closely separated wavelengths and offer increased flexibility in wavelength selection.

Alternatively, the method of excess fractions (EF) [26] can be applied with any set of N measurement wavelengths. The standard approach is to assume a value for the fringe order at one wavelength and hence determine the fringe orders at each of the other wavelengths using the set of measured fractional fringe values [6,27,28]. The correct solution is the one that gives fringe orders at all measurement wavelengths which are closest to integer values. One of the wavelength selection strategies applicable to EF is integer interferometry, in which the wavelengths are chosen to have integer values [23,29–34] or are integer multiples of some arbitrary common factor wavelength [35]. The common integer interferometry approaches take advantage of the aspects of number theory such that the unknown integer fringe orders can be calculated directly. In theory, the measurement range is set by the distance at which there is an integer multiple of each measurement wavelength, i.e., a wavelength coincidence. The distance where wavelength coincidence occurs is maximized when all measurement wavelengths have no common factors [34,35].

It has been shown that the maximum range achievable with EF is approximately equal to that predicted by GOMF given the same parameters [36,37]. Notably, a number of groups have presented results based on EF [2,23,28,34,37,38]. For example, in the work of Decker *et al.*, EF is used with non-integer measurement wavelengths in both a three- and four-wavelength MWI system [6]. Given the phase noise estimate of $2\pi/50$, the range should be extended by a factor of 5.89 for each additional measurement wavelength if the selections were optimum. However, a maximum increase by a factor of 3.73 in going to a four-wavelength system was reported.

In this work we present a new generic model which describes the relation between measurement wavelengths, phase noise, minimum achievable UMR, and the reliability of a MWI system. The model can be used to predict the performance of an MWI system *a priori*, without the need for initial experimental tests or numerical simulations. In addition, this model is not limited to integer wavelengths so that the performance is only affected by the uncertainty of the measured phase values. Furthermore, for a given phase noise, the reliability and measurement range can be quantified, and therefore used to adjust the measurement wavelengths in order to meet the specifications of a practical metrology system. Moreover, this method can be used to obtain a large UMR which, unlike GOMF, does not require closely separated measurement wavelengths. The results of this work can be used in any MWI application in order to improve the performance of the interferometer in terms of reliability and measurement range.

2. Method of EF

A. Background

The Michelson–Benoit method of EF [26] consists of a comparison of the fractional fringes for each wavelength. In MWI, the relation between the unknown fringe order m, the fractional fringe value ε, which is in the interval $[-0.5,\ \ 0.5]$, and the OPD for a set of N measurement wavelengths is given by

$$\mathrm{OPD} = u_0\lambda_0, \mathrm{OPD} = u_1\lambda_1, \ldots, \mathrm{OPD} = u_{N-1}\lambda_{N-1}, \quad (1)$$

where u is a dimensionless quantity $u_i = m_i + \varepsilon_i$ and λ_i is a measurement wavelength with $\lambda_0 < \lambda_1 < \ldots < \lambda_{N-1}$. The fringe order for the smallest wavelength m_0 must lie within the interval $[0, m_{0\,\mathrm{max}}]$, with $m_{0\,\mathrm{max}} = \mathrm{NINT}(\mathrm{UMR}/\lambda_0 - 1)$, where $\mathrm{NINT}(\cdot)$ is the nearest integer function. The fractional fringe values, $\varepsilon_0, \ldots, \varepsilon_{N-1}$, are obtained from the phase measurement at the N individual wavelengths. It is possible to derive another representation of Eq. (1) so that

$$\mathrm{OPD} = (M_{0i} + E_{0i})\Lambda_{0i}, \quad (2)$$

where $M_{0i} = m_0 - m_i$, $E_{0i} = \varepsilon_0 - \varepsilon_i$, and Λ_{0i} is the beat wavelength of the measurement wavelengths λ_i and λ_0 calculated by [18]

$$\Lambda_{0i} = \frac{\lambda_i \lambda_0}{\lambda_i - \lambda_0}. \tag{3}$$

In EF, a residual error is defined in order to find the correct integer fringe order m_0, which corresponds to the smallest measurement λ_0. To accomplish this, the fringe order at the ith wavelength, m_i, needs to be determined from

$$m_i = (m_0 + \varepsilon_0)\frac{\lambda_0}{\lambda_i} - \varepsilon_i. \tag{4}$$

The residual error, r_i, is then the difference between this calculated value for m_i and the nearest integer of this quantity, i.e.,

$$r_i(m_0) = \text{fract}\left((m_0 + \varepsilon_0)\frac{\lambda_0}{\lambda_i} - \varepsilon_i\right), \tag{5}$$

where fract is defined as the difference between a real value and its nearest integer, $\text{fract}(m_i) = m_i - \text{NINT}(m_i)$, and the LHS of Eq. (5) indicates that r_i is a function of m_0. In the general case, where N measurement wavelengths are used, an overall residual error, $R(m_0)$, can be determined as

$$R(m_0) = \sqrt{\sum_{i=1}^{N-1} |r_i(m_0)|^2}. \tag{6}$$

This function must be evaluated for all possible values of m_0 and the minimum overall residual error identifies the most likely solution for the fringe order m_0 and hence the OPD.

The residual error can also be expressed in terms of the beat wavelength as $(m_0 + \varepsilon_0)\lambda_0 = (M_{0i} + E_{0i})\Lambda_{0i}$, hence

$$r_i(m_0) = \text{fract}\left[-\frac{(m_0 + \varepsilon_0)\lambda_0}{\Lambda_{0i}} + E_{0i}\right]. \tag{7}$$

Equations (5) and (7) are equivalent. For convenience we introduce a dimensionless unit, the scaling factor $sf_i = \Lambda_{0i}/\lambda_0$, where $sf_i > 1$. Hence

$$r_i(m_0) = \text{fract}\left[-\frac{(m_0 + \varepsilon_0)}{sf_i} + E_{0i}\right]. \tag{8}$$

For further handiness, we remove the subscript of the smallest scaling factor $sf_{(N-1)}$ of the MWI system and refer to it simply as sf, so that $sf = sf_{(N-1)}$.

As the phase noise present in most interferometers has Gaussian statistics [39], the noise in the fractional fringe orders ε_0 and ε_i can be described with the standard deviations $\sigma_{\varepsilon 0}$ and $\sigma_{\varepsilon i}$, respectively. Given these considerations, an upper boundary of the uncertainty in the overall residual error R is given by

$$\sigma_R \le \sqrt{(N-1)(\sigma_{\varepsilon 0})^2 + \frac{1}{R^2}\sum_{i=1}^{N-1}(r_i(\varepsilon_0, \varepsilon_i))^2(\sigma_{\varepsilon i})^2}. \tag{9}$$

B. Two-Wavelength Interferometry with EF

For a two-wavelength system, the overall residual error is given by Eq. (8) with $i = 1$. The absolute value of the residual error is plotted in Fig. 1 for a scaling factor of 7.88 and $\varepsilon_0 = \varepsilon_1 = 0$. Selecting these particular values for the fractional fringe values implies that the correct solution is at $m_0 = 0$. For other fractional fringe values the graph is found to translate horizontally but has the same form, hence this position can be used without any loss of generality. The fract function generates a periodic result and therefore the residual error is also periodic. However, as m_0 is an integer, the sequence of possible values of the residual error generated for the case of $m_0 = 0, 1, 2, 3$, etc., is in most cases periodic. For the special case, where sf is an integer, the period of the sequence generated is equal to the period of the function $f(x)$ with $f(x) = \text{fract}(x/sf)$, where x is continuous. However, for the general case, where sf has a fractional part, the period of the sequence generated is larger than the period of $f(x)$. Notably, the period of the sequence is infinite, if sf is irrational. The positions of local minima depend on the number of wavelengths of λ_0 in the beat wavelength Λ_{01}, i.e., the scaling factor.

Any noise present in the interferometer generates a variation in the residual error values. Hence, all other residual error values except that at the correct solution must lie above the uncertainty generated by the noise. By increasing m_0, the first residual error value that lies within the uncertainty formed by the measurement noise determines the maximum unambiguous range. The residual error plot in Fig. 1 shows a number of critical points which are discussed below.

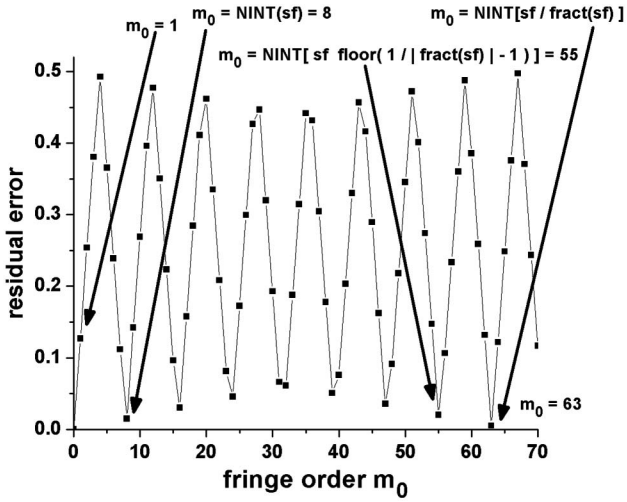

Fig. 1. Absolute residual error pattern for a two-wavelength system with a scaling factor of $sf = 7.88$.

1. *Residual Error at $m_0 = 1$*

The residual error, $r_1(m_0 = +1)$, must be above the uncertainty generated by the phase noise in order to range beyond $m_0 = 1$. When the scaling factor is large, $r_1(m_0 = +1)$ decreases closer to the uncertainty. From Eq. (8) and recalling that $\varepsilon_0 = \varepsilon_1 = 0$,

$$r_1(m_0 = +1) = \text{fract}\left[-\frac{1}{sf}\right]. \tag{10}$$

For the case where the uncertainty in the residual error can be estimated with Eq. (9), the separation in the residual error between potential fringe order solutions can be related to the probability of making a distance measurement beyond any particular range. Hence, provided the residual error values are separated by at least 6σ, there will be a 99.73% probability of being able to range beyond this point ($m_0 = 1$). Referring to Eq. (9), due to uncertainty in fractional fringe values (phase measurement noise), the noise in the overall residual error, σ_R, for the case of two measurement wavelengths is given by

$$\sigma_R \le \sqrt{(\sigma_{\varepsilon 0})^2 + (\sigma_{\varepsilon 1})^2},$$

which is reduced to $\sigma_R = \sqrt{2}\sigma_\varepsilon$, for the case $\sigma_\varepsilon = \sigma_{\varepsilon 0} = \sigma_{\varepsilon 1}$, where $\sigma\varepsilon_i$ is the standard deviation noise in the fractional fringe values. The fract function itself will not have any effect on noise propagation into the residual error as it is linear and continuous provided the magnitude of the noise term is as expected and $\ll 0.5$. Therefore, for a 6σ reliability in exceeding the range given by $m_0 = 1$, and thereby being able to range reliably up to the first beat wavelength

$$\left|\text{fract}\left[-\frac{1}{sf}\right]\right| \ge 6\sqrt{2}\sigma_\varepsilon, \tag{11}$$

where $|\cdot|$ denotes the absolute value. For $sf > 2$, $\text{fract}[1/sf] = 1/sf$, and therefore

$$sf \le 1/(6\sqrt{2}\sigma_\varepsilon). \tag{12}$$

This expression is directly equivalent to the noise limit developed in the GOMF approach ([Eq. (1) of [20]], with $\sigma_\varepsilon = \sigma_\phi/2\pi$. The possible values of m_0 within the measurement range are given as

$$m_0 = [0, \text{floor}(sf) - 1], \tag{13}$$

where $[x, y]$ indicates the range of possible values such that $m_0 \ge x$ and $m_0 \le y$ and floor$(\cdot)$ means round to the next smallest integer.

2. *Residual Error at the First Beat Wavelength*

Another notable value of the residual error is found for

$$m_0 = \text{NINT}(sf), \tag{14}$$

which corresponds to the value of the fringe order at λ_0, closest to the first beat wavelength. At this location, the first local minimum of the value of the residual error is to be expected and from Eqs. (8) and (14) is given by

$$r_1(m_0 = \text{NINT}(sf)) = \text{fract}\left[-(\text{NINT}(sf) + \varepsilon_0)\frac{1}{sf} + E_{01}\right]. \tag{15}$$

In the case plotted here (Fig. 1) with $\varepsilon_0 = \varepsilon_1 = 0$, using the relation $\text{fract}(x) = x - \text{NINT}(x)$, and considering the absolute value of the residual error

$$\begin{aligned} |r_1(m_0 = \text{NINT}(sf))| &= \left|\text{fract}\left[(\text{NINT}(sf))\frac{1}{sf}\right]\right| \\ |r_1(m_0 = \text{NINT}(sf))| &= \left|\text{fract}\left[\frac{(sf - \text{fract}(sf))}{sf}\right]\right| \\ &= \left|\text{fract}\left[\frac{1}{sf}\text{fract}(sf)\right]\right|, \end{aligned} \tag{16}$$

and because $sf > 1$, the value of the absolute residual error is given by

$$|r_1(m_0 = \text{NINT}(sf))| = \left|\frac{1}{sf}\text{fract}(sf)\right|. \tag{17}$$

It is possible to range beyond the distance given by $m_0 = \text{NINT}(sf)$, provided that a noise criterion similar to Eq. (12) is satisfied, which results in a 6σ reliability to range reliably with $m_0 > \text{NINT}(sf)$ of

$$\left|\frac{1}{sf}\text{fract}(sf)\right| \ge 6\sqrt{2}\sigma_\varepsilon,$$

so that

$$sf \le \frac{1}{6\sqrt{2}\,\sigma_\varepsilon}|\text{fract}(sf)|. \tag{18}$$

As fract returns values less than 0.5, it is clear by comparing Eqs. (12) and (18) that in order to range beyond the first beat wavelength [Eq. (12)], the scaling factor must be lower, i.e., the beat wavelength itself must be reduced.

C. Calculation of Local Minima with the Use of Continued Fractions

The total absolute residual error of a two-wavelength interferometer is calculated from Eqs. (6) and (8) and can also be expressed as

$$R(m_0) = \left|\mathrm{fract}\left[-\frac{m_0}{sf} - \left(\frac{\varepsilon_0}{sf} - (\varepsilon_0 - \varepsilon_1)\right)\right]\right| = \left|\mathrm{fract}\left[\frac{m_0}{sf} + f(\varepsilon_0, \varepsilon_1)\right]\right|, \quad (19)$$

where $f(\varepsilon_0, \varepsilon_1)$ is a constant offset calculated as

$$f(\varepsilon_0, \varepsilon_1) = \left(\frac{\varepsilon_0}{sf} - (\varepsilon_0 - \varepsilon_1)\right). \quad (20)$$

As shown in Eq. (19), it can be concluded that because fract$(\cdot)$ is periodic when applied to a continuous variable, the function $R(m_0)$ must be pseudoperiodic and the term $f(\varepsilon_0, \varepsilon_1)$ is only responsible for a constant offset. Every periodic function $g(x)$ with period T fulfills the property $g(x) = g(x+T)$, or $g(x) - g(x+T) = 0$. Similar properties apply also to $R(m_0)$, where the resulting pattern of the residual error has the pseudoperiodic properties $R(m_0) - R(m_0 + M) \approx 0$, where M is an integer. It should be noted that for the calculation of M it is sufficient to consider the case $f(\varepsilon_0, \varepsilon_1) = 0$, without loss of generality, because the offset $f(\varepsilon_0, \varepsilon_1)$ of Eq. (20) does not affect the periodicity of $R(m_0)$. Hence, at the correct fringe order $m_0 = m$ and the fringe order $m_0 = m + M$ the actual values of the residual error are given as $R(m) = 0$ and $R(m+M) \approx 0$, respectively, and therefore, the measurements remain only unambiguous if the possible values of m_0 are found in the interval $m_0 = [0, M-1]$. Therefore, if solely the minimum absolute values of $R(m_0)$ or the constant M is to be evaluated, the fractional fringe values can be set to $\varepsilon_0 = \varepsilon_1 = 0$, so that

$$R(m_0) = \left|\mathrm{fract}\left[\frac{m_0}{sf}\right]\right|. \quad (21)$$

Equation (21) implies that the correct value of m_0 is found to be $m_0 = 0$ (as $\varepsilon_0 = \varepsilon_1 = 0$) and the residual error at the neighboring fringe order $R(m_0 = 1) = 1/sf$ [see Eq. (10)]. Local minima of $R(m_0)$ occur at the location where the product $(m_0 \times 1/sf)$ is close to an integer, and the UMR is determined by the location for which fract$(M \cdot 1/sf) \approx 0$.

In Appendix A, a real number x_0 is developed with NICF (NICF), and the approximate values of x_0 as the ratio of two integers, $x_0 \approx s_k/q_k$, is discussed. The ratio s_k/q_k becomes a better approximation to x_0 as k increases. This technique can also be applied to the residual error defined in Eq. (21), using the development of the parameter $1/sf$ with NICF, as

$$x_0 = \frac{1}{sf} \approx \frac{s_k}{q_k} < 1, \quad (22)$$

so that the local minima of the residual error are found where $m_0 = |q_k|$ (i.e., m_0/sf becomes close to an integer). The corresponding values of the residual error at the points $m_0 = |q_k|$ are [from Eq. (A8) of Appendix A]

$$R(m_0) \geq \left|\frac{x_0}{p_{k+1}}\right|, \quad (23)$$

where the coefficients p_k are also derived from the continued fraction representation of $1/sf$ as defined in Appendix A. The s_k and q_k terms have a physical meaning in terms of MWI. The coefficients q_k represent the fringe order at the local minima in the residual error, and the coefficients s_k are the corresponding s_kth beat wavelength.

In the presence of an uncertainty in the residual error, σ_R [see Eq. (9)], the two-wavelength interferometer can be described by five parameters, Q, S, T, V, and W, which are defined as

$$S = |s_{k+1}| Q = |q_{k+1}| \quad \text{if } \{\exists! k \in N \,||1/V| < 6\sigma_R \leq |1/T|\}, \quad (24)$$

with

$$\begin{aligned} &T = p_{k+1}/x_0, V = p_{k+2}/x_0, W = |q_k| \quad \text{for } \mathrm{fract}(x_{k+1}) \neq 0 \\ &T = p_{k+1}/x_0, V = +\infty, W = 0 \quad \text{for } \mathrm{fract}(x_{k+1}) = 0, \end{aligned} \quad (25)$$

where the coefficients x_k are also derived from the NICF representation of $1/sf$ as defined in Appendix A and N denotes the set of natural numbers. The notation of Eqs. (24) and (25) provides a general expression for the residual error and the UMR of the two-wavelength interferometer and avoids the handling of the subscript k. The series of values q_k are a monotonically increasing set of integer fringe orders, and the minimum value of the residual error up to any particular q_k is given by $|\mathrm{fract}[|q_k|/sf]|$. The limiting UMR in any particular situation can be found by comparing the uncertainty in the residual error (which depends on the phase noise of the interferometer) with the minimum residual error values, which decrease as m_0 increases. Therefore, the approach taken in Eq. (24) is to examine each interval $m_0 = |q_k|$ and $m_0 = |q_{k+1}|$ and refer to the absolute value of the residual error at either end of the interval using the terms T and V where $|\mathrm{fract}[|q_k|/sf]| = |1/T|$ and $|\mathrm{fract}[|q_{k+1}|/sf]| = |1/V|$, respectively. The limiting UMR is found when the values $1/T$ and $1/V$ bracket the minimum residual error $6\sigma_R$.

To highlight these properties, the exemplary case of a two-wavelength interferometer with $sf = 7.88$ is shown in Fig. 1 and is replotted in Fig. 2 together with its estimated lower boundary. In addition, Fig. 2 also highlights the values of $1/T$ and $1/V$ for the exemplary cases that correspond to $k = 0$, $k = 1$, and $k = 2$.

Given these facts, it is possible to identify the fringe order, $Q - W$, at which the minimum value of the residual error reduces to less than the uncertainty in the residual error derived from a practical system with phase noise. Similar to the example in Appendix B, a lower bound of the residual error can be expressed as

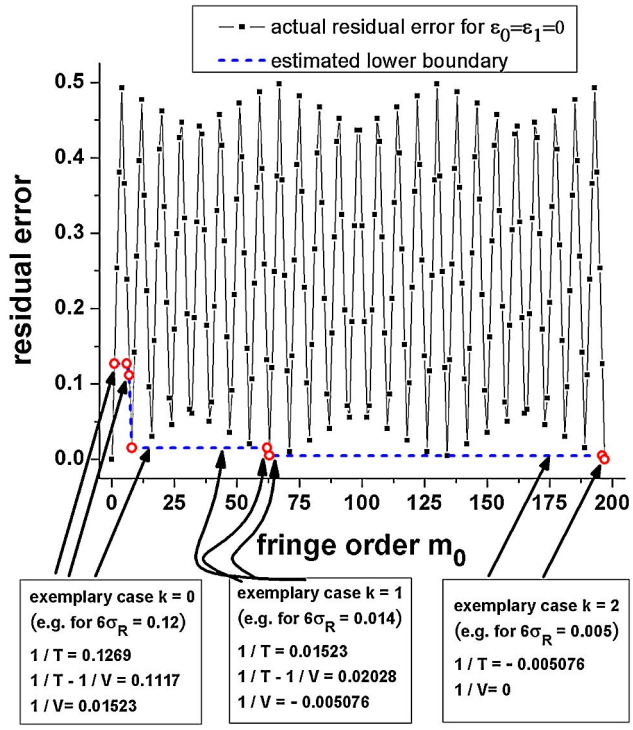

Fig. 2. (Color online) Absolute residual error and the estimated lower boundary for a two-wavelength system with a scaling factor of $sf = 7.88$. The minimum value of the residual error is highlighted for the exemplary cases with $k = 0$, $k = 1$, and $k = 2$.

$$|R(m_0)| \geq \begin{cases} \left|\frac{1}{T}\right| & \text{for } m_0 = [|q_k|, (|q_{k+1}| - |q_k|) - 1] \\ \left|\frac{1}{T}\right| - \left|\frac{1}{V}\right| & \text{for } m_0 = [(|q_{k+1}| - |q_k|), (|q_{k+1}| - 1)], \\ \left|\frac{1}{V}\right| & \text{for } m_0 = |q_{k+1}| \end{cases} \tag{26}$$

where $|1/T|$ is assumed to be always above the noise floor from Eq. (24). Given these considerations, the parameters Q and W, defined in Eqs. (24) and (25), can be used to describe the resulting UMR as

$$\text{UMR} = (Q - W)\lambda_0, \tag{27}$$

so that the possible values of the fringe order m_0 are in the interval

$$m_0 = [0, (Q - W - 1)]. \tag{28}$$

The use of the parameters Q, S, T, V, and W has two significant advantages compared to conventional algorithms that only make use of the measurement wavelengths or the scaling factor. First, the calculation of these parameters takes into account the interdependence of phase noise, measurement wavelengths, and UMR, which is actually not the case for conventional approaches where the scaling factor is just given by the measurement wavelengths used in the system. This is of particular importance if it is desired to expand the measurement range beyond the largest beat wavelength, a region where currently established techniques fail to predict actual UMRs as well as the reliability of the measurement. Second, the use of these parameters also allows a comparison of various choices of measurement wavelengths in terms of their performance. It is possible to calculate different sets of measurement wavelengths that create the same parameters Q, S, T, V, and W and are therefore equivalent in terms of performance. On the other hand, it is interesting to show which sets of wavelengths correspond to the same value for Q and W, i.e., are able to range over the same UMR without compromising the reliability.

This theory can be used to define the UMR of a two-wavelength interferometer beyond any of the critical points identified in Fig. 1, corresponding to increasing values of k, depending on the noise level in the interferometer. An analogous approach for measurements beyond the first beat wavelength ($k \leq 2$) has been reported by de Groot [25] in which the UMR may initially be assumed to be $sf/|\text{fract}(sf)|\lambda_0$, when the noise criterion [Eq. (18)] is fulfilled. However, after detailed analysis it has been found that the measurement range is given by Eq. (27), in which W may not be zero, and the noise criterion expressed in Eqs. (24) and (25) must be satisfied.

For convenience, and for reasons that become clear later, the minimum value of the residual error within the UMR of the two-wavelength interferometer is defined as

$$\psi_{sf} = \left|\frac{1}{T}\right|. \tag{29}$$

The noise criteria in Eqs. (18) and (24) offer a vast number of solutions when the interferometer wavelengths can be chosen to define a desired value of sf. This is further investigated for a two-wavelength interferometer with a fixed wavelength $\lambda_0 = 405\,\text{nm}$ and the second wavelength λ_1 in the interval 532 to 780 nm ($sf = [2.08, 4.189]$). The resulting UMR for an interferometer with a phase noise of $1/1000$th of a fringe is shown in Fig. 3. The special case of $k_{\max} = 2$ (with the same range as the de Groot extension), and the case of an unbounded value of $k_{\max}$ is shown in Figs. 3(a) and 3(b), respectively.

It is clear from Fig. 3(b) that allowing the system to range over multiple beat wavelengths can be tolerated for a multitude of additional selections for λ_1 and will give a high UMR and reliability in calculating the fringe orders. It is interesting to notice that in contrast to other approaches [24,25], when using EF, any extended UMR can be achieved without modifying the data processing algorithms.

3. Multiwavelength Interferometry

A. Three-Wavelength Interferometry with EF

In many applications, the UMR of a conventional two-wavelength interferometer is not large enough. The addition of a further measurement wavelength

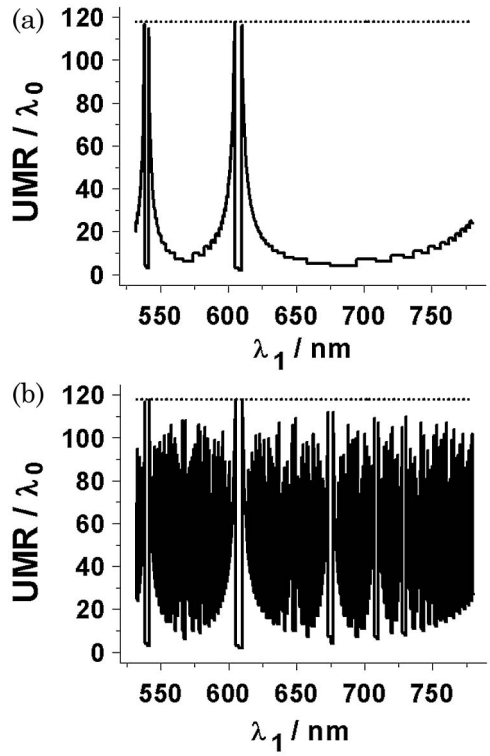

Fig. 3. Extended UMR (solid curve) of a two-wavelength system with $\lambda_0 = 405\,\text{nm}$ and a phase noise of 1/1000th of a fringe with (a) $k_{\max} = 2$, and (b) with an unbounded value of $k_{\max}$. The corresponding noise limited value is indicated by the dotted line.

enables a larger measurement range. Considering a three-wavelength system with $\lambda_0 < \lambda_1 < \lambda_2$, $sf = \Lambda_{02}/\lambda_0$, $c/\lambda_0 > c/\lambda_1 > c/\lambda_2$, and $f_0 > f_1 > f_2$, where c is the speed of light and f_0, f_1, and f_2 are the frequencies corresponding to the wavelengths λ_0, λ_1, and λ_2. The beat frequencies F_{0n} are given by $F_{02} = f_0 - f_2$ and $F_{01} = f_0 - f_1 = (1-\alpha)F_{02}$, where α is a dimensionless factor and $0 < \alpha < 1$. Hence, f_2 is given as $f_1 = f_2 + \alpha F_{02}$. The relation between the corresponding beat wavelengths and the scaling factors is given by $\Lambda_{01} = \Lambda_{02}/(1-\alpha)$ and $sf_{01} = sf_{02}/(1-\alpha)$, respectively. According to Eqs. (6) and (7), the total residual error is the sum of the two individual error components.

$$R(m_0) = \sqrt{\left|\text{fract}\left(-\frac{L(m_0)}{\Lambda_{02}} + E_{02}\right)\right|^2 + \left|\text{fract}\left(\frac{L(m_0)}{\Lambda_{01}} + E_{01}\right)\right|^2} = \sqrt{\left|\text{fract}\left(-\frac{m_0+\varepsilon_0}{sf} + (\varepsilon_0 - \varepsilon_2)\right)\right|^2 + \left|\text{fract}\left(-(1-\alpha)\frac{m_0+\varepsilon_0}{sf} + (\varepsilon_0 - \varepsilon_1)\right)\right|^2}, \quad (30)$$

where $L(m_0) = (m_0 + \varepsilon_0)\lambda_0$ and $sf = sf_{02}$, and from Eq. (9) an upper bound of the standard deviation of the residual error is given as

$$\sigma_R \le \sqrt{(\sigma_{\varepsilon 0})^2 + \frac{1}{R^2}\sum_{i=1}^{2}(r_i(\varepsilon_0, \varepsilon_i))^2(\sigma_{\varepsilon i})^2}, \quad (31)$$

which is reduced to $\sigma_R \le \sqrt{3}\sigma_\varepsilon$ for $\sigma_\varepsilon = \sigma_{\varepsilon 0} = \sigma_{\varepsilon 1} = \sigma_{\varepsilon 2}$.

1. *Residual Error at $m_0 = \pm 1$*

Subsection 2.B.1, describes that the EF solver must be able to distinguish between the correct value of the residual error and values of the residual error which are one fringe order apart. A similar criterion is also valid for the three-wavelength interferometer; from Eq. (30) and recalling that $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$, the residual error at $m_0 = +1$ with $sf > 2$ is equal to

$$R(m_0 = +1) = \frac{1}{sf}\sqrt{1 + (1-\alpha)^2}.$$

For convenience, we define a dimensionless quantity δ as

$$\delta = \frac{1}{sf}\sqrt{1 + (1-\alpha)^2}. \quad (32)$$

Similarly to Subsection 2.B.1, a noise criterion for 6σ reliability is defined as

$$\delta \ge 6\sigma_R, \quad (33)$$

where σ_R is the standard deviation of the total residual error, defined in Eq. (31). If this noise criterion is fulfilled, the UMR exceeds the range given by $m_0 = 1$; hence, it is possible to range reliably up to the first beat wavelength.

2. *Residual Error at $m_0 = NINT(nS_{sf}sf)$*

As it was for a two-wavelength interferometer, local minima of the residual error occur at integer multiples of the smallest beat wavelength. However, in the case of a three-wavelength interferometer, the significant locations of low values of the residual error are found at integer multiples, n, of the UMR of an interferometer utilizing only the smallest and largest measurement wavelength. The overall residual error of the three-wavelength system still has local minima at multiples of the UMR of the two-wavelength interferometer. Subsection 2.B investigated the residual error of the two-wavelength interferometer in Eq. (21) using the NICF development of $x_0 = 1/sf$, such that the residual error could be described with five parameters, Q_{sf}, S_{sf}, T_{sf}, V_{sf}, and W_{sf}, where the subscript sf refers to the continued fractions development of $x_0 = 1/sf$. The subscript notation is introduced at this point in order to avoid any confusion of developed continued fractions of other constants. The local minima of the residual error are found where

$$m_0 = \text{NINT}(nS_{sf}sf) = nS_{sf}sf - \text{fract}(nS_{sf}sf),$$

where n is an integer ($n = 1, 2, 3, \ldots$). The corresponding residual error for the fringe order m_0, at the distance $L(m_0) = (\mathrm{NINT}(nS_{sf}sf) + \varepsilon_0)\lambda_0$, for $\varepsilon_0 = \varepsilon_1 = 0$, is given by

$$R(m_0 = \mathrm{NINT}(nS_{sf}sf) = \sqrt{\rho[nS_{sf}, sf]^2 + |\mathrm{fract}[(1-\alpha)nS_{sf} - (1-\alpha)\rho[nS_{sf}, sf]]|^2}, \tag{34}$$

where $\rho[nS_{sf}, sf] = \mathrm{fract}(nS_{sf}sf)/sf$. The residual errors with low absolute value are found to have a small value of $\rho[nS_{sf}, sf]$, and $(1-\alpha)nS_{sf}$ is close to an integer value, so that

$$|\mathrm{fract}[(1-\alpha)nS_{sf}] - (1-\alpha)\rho[nS_{sf}, sf]| < 0.5.$$

And therefore, using $\mathrm{fract}(x \pm y) = \mathrm{fract}(x) \pm \mathrm{fract}(y)$, Eq. (34) can be expressed as

$$R(m_0 = \mathrm{NINT}(nS_{sf}sf) = \sqrt{\rho[nS_{sf}, sf]^2 + (\mathrm{fract}[n(1-\alpha)S_{sf}] - (1-\alpha)\rho[nS_{sf}, sf])^2}.$$

When the product $S_{sf}sf$ is approximately an integer, i.e., taking the worst case, $\rho[nS_{sf}, sf] \approx 0$ and the previous expression reduces to

$$R(m_0 = \mathrm{NINT}(nS_{sf}sf) = |\eta[n, (1-\alpha)S_{sf}]|, \tag{35}$$

with

$$\begin{aligned}\eta[n, (1-\alpha)S_{sf}] &= \mathrm{fract}[n(1-\alpha)S_{sf}] \\ &= \mathrm{fract}[n\,\mathrm{fract}[(1-\alpha)S_{sf}]] \\ &= \mathrm{fract}[n/sf_\alpha],\end{aligned} \tag{36}$$

where

$$(sf)_\alpha = 1/\mathrm{fract}[(1-\alpha)S_{sf}]. \tag{37}$$

The quantity sf_α can be considered as an equivalent scaling factor with $|(sf)_\alpha| \geq 2$.

As discussed in Subsection 2.C, the UMR of a two-wavelength interferometer is limited by phase noise. For instance, a two-wavelength system with $sf = 3.553$, $S_{sf} = 9$, $V_{sf} = 154.48$, $Q_{sf} = 32$, $W_{sf} = 7$ has an UMR of $25\lambda_0$ and a minimum reliability of 99.73% when the uncertainty of the fractional fringe values σ_ε is ≤1/290th of a fringe so that the uncertainty in R is $6\sigma_R \leq |1/Tsf| = 1/33.51$.

The same two-wavelength system would be able to achieve a minimum UMR of $135\lambda_0$ if the phase noise of the system was significantly reduced ($\sigma_\varepsilon \approx$ 1/1300th of a fringe). However, some interferometers do not provide these phase noise requirements. In such situations, the residual error term $\rho(nS_{sf}, sf)$ of the two-wavelength system is in the order of the noise floor.

Notably, Eq. (35) shares similarities with Eq. (8) and indicates that a higher UMR is achievable with the introduction of a third measurement wavelength. As for the two-wavelength system, it is possible to develop a nearest integer continued fraction for the quantity $1/sf_\alpha \approx s_k/q_k$ in order to determine the location $n = q_k$, where the significant low values of η occur.

The residual error of Eq. (35) is therefore similar to the case of a two-wavelength interferometer [see Eqs. (21) and (22)], and the equivalent scaling factor is used for the NICF

$$x_0 = \frac{1}{sf_\alpha} \approx \frac{s_k}{q_k} < 1.$$

An exemplary case is a three-wavelength interferometer with $\lambda_0 = 1530\,\mathrm{nm}$ and $\lambda_2 = 1608.2489\,\mathrm{nm}$, $sf = 20.553$ having an uncertainty of the fractional fringe values σ_ε is ≤1/520th so that the noise floor $\sigma_R \leq 1/50$th. It can be shown that a UMR $\approx 1000\lambda_0$ could be obtained and a vast number of choices for λ_1 provide a practically achievable UMR $> 500\lambda_0$.

Another example of the residual error of a three-wavelength system with $sf = 25.4$ and $\alpha = 7/13$ and $\sigma_R = 1/50$ is shown in Fig. 4. The relevant coefficients are $sf = 25.4$, $S_{sf} = 1$, $V_{sf} = 63.5$, $Q_{sf} = 25$, and $W_{sf} = 1$. The influence of the center wavelength is described with the parameter $1/(sf)_\alpha = 6/13$. Similar to the case of the NICF with $x = 1/sf$, it is possible to develop $x = 1/sf_\alpha$ and use the resulting parameters Q_α, S_α, T_α, V_α, and W_α to analyze the behavior of η_α. In line with the previously introduced notation, the subscript α refers to the continued fractions development of $x_0 = 1/sf_\alpha$. For $1/sf_\alpha = 6/13$, the resulting coefficients are $T_\alpha = 13$, $V_\alpha \to +\infty$, $W_\alpha = 0$, and $Q_\alpha = 13$.

Consequently, the UMR is equal to

$$\mathrm{UMR} = \mathrm{floor}[(Q_\alpha - W_\alpha)S_{sf}sf - W_{sf}]\lambda_0, \tag{38}$$

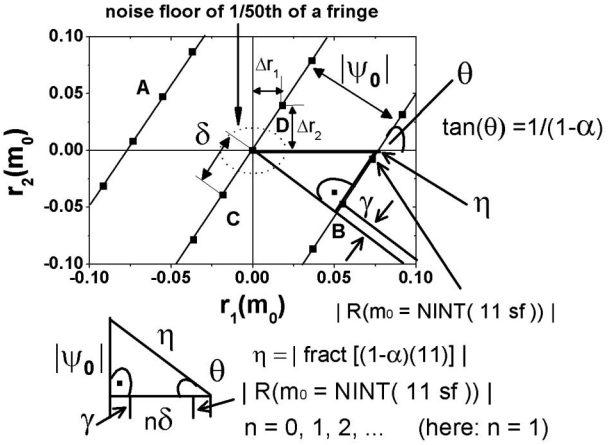

Fig. 4. Values of the residual error for various values of m_0 for $sf = S_{sf}sf = 25.4$ and $\alpha = 7/13$ with $(Q_\alpha - W_\alpha) = 13$. The residual error shown corresponds to the fractional fringe orders $\varepsilon_0 = -0.3706$, $\varepsilon_1 = 0.4199$, and $\varepsilon_2 = -0.3245$.

with possible values for m_0 in the interval

$$m_0 = [0, \text{floor}[(Q_\alpha - W_\alpha)S_{sf}sf - W_{sf}] - 1]. \quad (39)$$

Hence, the exemplary case of the three-wavelength system with $sf = 25.4$ and $\alpha = 7/13$ and $\sigma_R = 1/50$ has a UMR equal to

$$\text{UMR} = \text{floor}[(13 - 0) \times 1 \times 25.4 - 1]\lambda_0 = 329\lambda_0.$$

As for the two-wavelength system, reliable measurements can only be carried out when the noise criterion of Eq. (33) is fulfilled. This can be seen in Fig. 4 by examining the diagonal line through $r_1(m_0) = r_2(m_0) = 0$, along which m_0 increases monotonically and the difference in residual error values between consecutive points is given by δ. However, from Fig. 4 it can also be seen that there are other points in neighboring diagonals that are close to the correct solution, indicated as A and B in Fig. 4. Depending on the relative magnitudes of ψ_0, δ, and γ (which in turn depend on α, sf, and m_0), the potential solutions A and B may be closer to the correct answer than C and D (which lie along the diagonal through 0,0). The fringe order at A and B is not necessarily one of the set of points defined by $m_0 = \text{NINT}(nS_{sf}sf)$, which can occur in two circumstances: when the combination of δ and γ mean that the next point closest to the beat is in fact closer to the correct solution and when $S_{sf} > 1$.

The corresponding absolute value of $R(m_0)$ can be estimated based on a geometrical analysis; any value of the residual error can be represented with the quantities γ, δ, and ψ_0, and introducing two integer variables t and b to indicate the position along a diagonal of increasing m_0 and the neighboring diagonals, respectively. Therefore, for $t = 4$, $m_0 = 4$ and the residual error is 4δ; and the residual error difference between neighboring diagonals (increasing b) is ψ_0. There is an additional complexity as the location of the integer fringe orders on neighboring diagonals is not necessarily the same, and this is identified as γ (see Fig. 4).

Therefore, as a function of b and t, the residual error is defined as

$$|R(b,t)| = \begin{cases} \sqrt{(t\delta)^2} & \text{for } b = 0 \\ \sqrt{(b\psi_0)^2 + (\gamma(b) - t\delta)^2} & \text{for } b \neq 0 \end{cases}, \quad (40)$$

where b and t are integer ($b = 0, \pm1, \pm2, \ldots$ and $t = 0, \pm1, \pm2, \ldots$), $\gamma(b = 0) = 0$, and ψ in general is given by

$$\psi(b) = \frac{\eta[n, 1/sf_\alpha]}{\sqrt{1 + 1/\tan^2\theta}}, \quad (41)$$

where $\tan\theta = 1/(1-\alpha)$, $\psi(0) = 0$, and $\psi_0 = \psi(b = \pm1)$. The two lowest possible values of ψ are denoted as ψ_0 and calculated as

$$\psi_0 = \pm\frac{1}{T_\alpha}\frac{1}{\sqrt{1 + (1-\alpha)^2}}, \quad (42)$$

where T_α has been obtained from the developed NICF of $1/sf_\alpha$ so that $1/T_\alpha$ is the minimum possible absolute value of η, defined by Eqs. (36) and (42), and n_0 is the corresponding value for n [see Eq. (36)].

One of the minimum nonzero values of the residual error of Eq. (40) is found at the location $t = \pm1$ with $|R(t = \pm1)| = \delta$. The interferometer has to distinguish between the value $R(t = 0)$ and $|R(t = \pm1)| = \delta$, which leads to the previously developed noise criteria of Eq. (33). In the same way, the second minimum absolute value of the residual error of Eq. (40) is found to be

$$|R| = \begin{cases} \sqrt{\psi_0^2 + \gamma^2} & \text{for } \gamma \leq \delta/2 \\ \sqrt{\psi_0^2 + (\delta - \gamma)^2} & \text{for } \gamma > \delta/2 \end{cases}, \quad (43)$$

where the quantity γ can be found in the range $\gamma = [0, \delta]$. The absolute value of γ is dependent on δ and $R(\text{NINT}(zsf))$, which are affected by the choice of α and sf (z is an integer). In best case scenarios, the distance γ is equal to $\delta/2$. In worst case scenarios, the distance γ is close to zero so that the absolute value of the residual error is equal to $|R(m_0)| \approx \psi_0$. Consequently, a noise criterion based on Eqs. (33) and (43) can be defined as

$$|\psi_0| \geq 6\sigma_R, \quad (44)$$

which together with the criterion in Eq. (33) ensures high reliability for all absolute values of the residual error.

Therefore, similar to the case of Eqs. (24) and (25), it is possible to define the five parameters, $Q_\alpha, S_\alpha, T_\alpha, V_\alpha$, and W_α for the case of $x = 1/sf_\alpha$ as

$$S_\alpha = |s_{k+1}| Q_\alpha = |q_{k+1}| \quad \text{if } \{\exists! k \in N || 1/V_\alpha| < 6\sigma_R\sqrt{1 + (1-\alpha)^2} \leq |1/T_\alpha|\}, \quad (45)$$

with

$$\begin{matrix} T_\alpha = p_{k+1}/x_0, V_\alpha = p_{k+2}/x_0, W_\alpha = |q_k| & \text{for fract}(x_{(k+1)}) \neq 0 \\ T_\alpha = p_{k+1}/x_0, V_\alpha = +\infty, W_\alpha = 0 & \text{for fract}(x_{(k+1)}) = 0. \end{matrix}, \quad (46)$$

so that Eq. (44) is fulfilled. The factor $\sqrt{1+(1-\alpha)^2}$ is introduced because the nearest integer continuous fraction development is for η rather than ψ [see Eq. (41) and (42)].

B. Four-Wavelength Interferometry with EF

Similar to the case of three measurement wavelengths, a four-wavelength interferometer can be described with the parameters sf, α_1, and α_2, where $0<\alpha_2<\alpha_1<1$. Hence, the corresponding beat wavelengths are given by $\Lambda_{01}=\Lambda_{03}/(1-\alpha_1)$ and $\Lambda_{02}=\Lambda_{03}/(1-\alpha_2)$, having, for the case of $\varepsilon_0=\varepsilon_1=\varepsilon_2=\varepsilon_3=0$, an overall residual error of

$$R(m_0)=\sqrt{\begin{array}{l}\left|\mathrm{fract}\left(\frac{m_0}{sf}\right)\right|^2+\left|\mathrm{fract}\left((1-\alpha_1)\frac{m_0}{sf}\right)\right|^2\\ +\left|\mathrm{fract}\left((1-\alpha_1)\dfrac{m_0}{sf}\right)\right|^2\end{array}}. \tag{47}$$

Similar to the case of the two- and three-wavelength systems, it is possible to evaluate the residual error at $m_0=1$ and define a dimensionless quantity δ:

$$\delta=\frac{1}{sf}\sqrt{1+(1-\alpha_1)^2+(1-\alpha_2)^2}, \tag{48}$$

which needs to fulfill a noise criterion defined as

$$\delta\geq 6\sigma_R. \tag{49}$$

The upper boundary of σ_R can be calculated from Eq. (9) as

$$\sigma_R\leq\sqrt{3(\sigma_{\varepsilon 0})^2+\frac{1}{R^2}\sum_{i=1}^{3}(r_i(\varepsilon_0,\varepsilon_i))^2(\sigma_{\varepsilon i})^2}, \tag{50}$$

which is reduced to $\sigma_R\leq 2\sigma_\varepsilon$ for $\sigma_\varepsilon=\sigma_{\varepsilon 0}=\sigma_{\varepsilon 1}=\sigma_{\varepsilon 2}=\sigma_{\varepsilon 3}$. At distances, which are equal to an integer multiple of the UMR of the two-wavelength system, given by $m_0=\mathrm{NINT}(nS_{sf}sf)$, the residual error of the four-wavelength system is found to be

$$\begin{aligned}&R(m_0=\mathrm{NINT}(nS_{sf}sf))\\&=\sqrt{\begin{array}{l}\rho[nS_{sf},sf]^2\\ +(\eta[n,1/sf_{\alpha 2}]-(1-\alpha_2)\rho[nS_{sf},sf])^2,\\ +(\eta[n,1/sf_{\alpha 1}]-(1-\alpha_1)\rho[nS_{sf},sf])^2\end{array}}\end{aligned} \tag{51}$$

where $1/sf_{\alpha 1}=\mathrm{fract}[(1-\alpha_1)S_{sf}]$ and

$$1/sf_{\alpha 2}=\mathrm{fract}[(1-\alpha_2)S_{sf}]. \tag{52}$$

For convenience, it is assumed that at the location $m_0=\mathrm{NINT}(nS_{sf}sf)$, the influence of $\rho(nS_{sf},sf)$ is negligible $\rho(nS_{sf},sf)\approx 0$, so that Eq. (51) is reduced to

$$\begin{aligned}&R(m_0=\mathrm{NINT}(nS_{sf}sf)\\&=\sqrt{(\eta[n,1/sf_{\alpha 2}])^2+(\eta[n,\beta_1/sf_{\alpha 2}])^2}\\&=\sqrt{(\mathrm{fract}[n/sf_{\alpha 2}])^2+(\mathrm{fract}[\beta_1 n/sf_{\alpha 2}])^2},\end{aligned} \tag{53}$$

with $\beta_1=(1-\alpha_1)/(1-\alpha_2)$ and $0<\beta_1<1$ and both residual error terms are expressed in terms of $sf_{\alpha 2}$. Equation (53) shares similarities with Eq. (30). Similar to the case of the three-wavelength system, the factor $1/sf_{\alpha 2}$ can be developed with NICF in order to estimate the UMR of the system. The resulting system performance is dependent on the parameter $\psi_{\alpha 2}$:

$$\psi_{\alpha 2}=\frac{\eta[n_0,1/sf_{\alpha 2}]}{\sqrt{1+(1-\alpha_2)^2}}, \tag{54}$$

which is calculated in the same way as ψ_0 of Eq. (42), using $m_0=\mathrm{NINT}(n_0S_{sf}sf)$, and has to fulfill a noise criterion, defined as

$$|\psi_{\alpha 2}|\geq 6\sigma_R. \tag{55}$$

The presence of the additional term $\eta[n,\beta_1/sf_{\alpha 2}]$ is of particular importance, when at a multiple of the UMR of the three-wavelength interferometer with the measurement wavelengths λ_0, λ_2, and λ_3. At this location $n=S_\alpha$ and the product $(1/sf_{\alpha 2}\times n)$ is close to an integer value. In accordance to Subsection 3.A.2, the value of the residual error at

$$n=\mathrm{NINT}(c_1S_{\alpha 2}sf_{\alpha 2}), \tag{56}$$

where c_1 is an integer $(c_1=1,2,3,\ldots)$ needs to be investigated and represents a multiple of the UMR of the three-wavelength interferometer. This location is equivalent to the UMR of the three-wavelength interferometer having a low value of the residual error term $\eta[n,1/sf_{\alpha 2}]$, which vanishes in the noise floor. The residual error of Eq. (53) is therefore written as

$$\begin{aligned}&R(n=\mathrm{NINT}(c_1S_{\alpha 2}sf_{\alpha 2}))\\&=\sqrt{(\mathrm{fract}[n/sf_{\alpha 2}])^2+(\mathrm{fract}[\beta_1 n/sf_{\alpha 2}])^2}\\&=\sqrt{\begin{array}{l}(\mathrm{fract}[c_1S_{\alpha 2}-\rho[c_1S_{\alpha 2},sf_{\alpha 2}]])^2,\\ +(\mathrm{fract}[\beta_1c_1S_{\alpha 2}-\beta_1\rho[c_1S_{\alpha 2},sf_{\alpha 2}]])^2\end{array}}\end{aligned} \tag{57}$$

where $\rho[c_1S_{\alpha 2},sf_{\alpha 2}]=\mathrm{fract}(c_1S_{\alpha 2}sf_{\alpha 2})/sf_{\alpha 2}$ and $\beta_1=(1-\alpha_1)/(1-\alpha_2)$ with $\beta_1<1$. When the product $S_{\alpha 2}sf_{\alpha 2}$ is approximately an integer, i.e., taking the worst case, the value of r is small, $\rho[c_1S_{\alpha 2},sf_{\alpha 2}]\approx 0$, and Eq. (57) reduces to

$$\begin{aligned}&R(n=\mathrm{NINT}(c_1S_{\alpha 2}sf_{\alpha 2}))\\&\approx\sqrt{(\mathrm{fract}[c_1S_{\alpha 2}])^2+(\mathrm{fract}[\beta_1c_1S_{\alpha 2}])^2}\\&\approx|\mathrm{fract}[\beta_1c_1S_{\alpha 2}]|,\end{aligned} \tag{58}$$

where $sf_{\beta 1}$ is the equivalent scaling factor, similar to the scaling factor defined in Eq. (53) and is calculated as

$$sf_{\beta 1} = 1/\text{fract}[\beta_1 S_{\alpha 2}]. \tag{59}$$

Then $sf_{\beta 1}$ can be developed with NICF, so that the result in Eqs. (58) and (59) is similar to Eq. (41), having $\tan\theta = \beta_1$, $\eta = |\text{fract}[c_1/sf_{\beta 1}]|$. Hence a parameter $\psi_{\beta 1}$ can be defined as

$$\psi_{\beta_1} = \frac{\eta[c_1, 1/sf_{\beta 1}]}{\sqrt{1+\beta_1^2}}, \tag{60}$$

which has to fulfill the noise criterion for all possible values of k so that

$$|\psi_{\beta 1}| \geq 6\sigma_R. \tag{61}$$

It becomes clear that in order to fulfill the noise criterion, the parameter β_1 may need to be adjusted, i.e., the actual value of the second or third measurement wavelength may have to be changed.

As in Subsection 3.A.2, it is possible to treat the case of $x = 1/sf_{\beta 1}$ by the NICF [see Eqs. (24) and (25)]. Therefore, it is possible to define the five parameters, $Q_{\beta 1}$, $S_{\beta 1}$, $T_{\beta 1}$, $V_{\beta 1}$, and $W_{\beta 1}$ for the case of $x = 1/sf_{\beta 1}$ as

$$S_\beta = |s_{k+1}| Q_\beta = |q_{k+1}|$$
$$\text{if } \{\exists! k \in N || 1/V_\beta| < 6\sigma_R\sqrt{1+\beta_1^2} \leq |1/T_\beta|\}, \tag{62}$$

with

$$T_\beta = p_{k+1}/x_0, V_\beta = p_{k+2}/x_0, W_\beta = |q_k| \text{for fract}(x_{(k+1)}) \neq 0$$
$$T_\beta = p_{k+1}/x_0, V_\beta = +\infty, W_\beta = 0 \text{for fract}(x_{(k+1)}) = 0, \tag{63}$$

so that Eq. (61) is fulfilled.

C. General MWI with EF

Similar to the case of the three- and four-wavelength system, it is possible to describe a general multi-wavelength system having the measurement wavelengths $\lambda_0 < \lambda_1 < \ldots < \lambda_{N-1}$, with the parameter $sf = \Lambda_{0(N-1)}/\lambda_0$, and the dimensionless factors with $0 < \alpha_{N-2} < \ldots < \alpha_2 < \alpha_1 < 1$.

Similar to the previous sections, dimensionless quantities δ and ψ_{sf} are defined as

$$\delta = \frac{1}{sf}\sqrt{1 + \sum_{i=1}^{N-2}(1-\alpha_i)^2}, \tag{64}$$

$$\psi_{sf} = \left|\frac{1}{T_{sf}}\right|,$$

where ψ_{sf} has been defined in Eq. (29).

The parameters $\psi_{\alpha(N-2)}, \psi_{\beta(N-3)}, \ldots, \psi_{\beta 1}$ can be calculated from the constants

$$sf_\alpha = \text{fract}[(1-\alpha_{N-2})S_{sf}],$$
$$sf_{\beta(N-3)} = \text{fract}[\beta_{(N-3)} S_\alpha],$$
$$sf_{\beta(N-2)} = \text{fract}[\beta_{(N-2)} S_{\beta_{(N-3)}}], \ldots sf_{\beta 1} = \text{fract}[\beta_1 S_{\beta_2}],$$

as

$$\psi_\alpha = \frac{1}{T_\alpha}\frac{1}{\sqrt{1+(1-\alpha_{N-2})^2}},$$

$$\psi_{\beta_n} = \frac{1}{T_{\beta n}}\frac{1}{\sqrt{1+(\beta_n)^2}},$$

with

$$\beta_n = \frac{1-\alpha_n}{1-\alpha_{n+1}}, \tag{65}$$

where $1/T_\alpha$ and $1/T_\beta$ have been obtained from the continued fraction development of Eqs. (45) and (62), respectively.

Reliable measurements are carried out if the corresponding values for δ and ψ fulfill the noise criterion

$$\min\{\delta, \psi_{sf}, \psi_{\alpha(N-2)}, \psi_{\beta(N-3)}, \ldots, \psi_{\beta 1}\} \geq 6\sigma_R. \tag{66}$$

An upper boundary of the parameter σ_R can be calculated from Eq. (9).

4. Previous Developments in the Literature

The performance of MWI depends strongly on the choice of measurement wavelengths. The model for EF presented in this work allows measurement wavelengths to be identified, which are more favorable than others in terms of the performance of the interferometer. It has also been shown that optimum wavelength selections previously derived for beat wavelength techniques are a subset of the theory developed, as reported by Towers *et al.* [20]. Nevertheless, the use of EF allows significantly more freedom in the choice of sets of measurement wavelengths while delivering similar performance to optimum, geometric series beat wavelength approaches.

This section applies the framework developed in the previous sections to discuss a variety of wavelength selections that are commonly used in MWI.

A. Number Theoretical Approaches

Number theoretical approaches, also known as integer interferometric methods, are known to have a UMR equal to the distance where wavelength coincidence occurs. The principle of integer interferometry has been applied to EF [24], but can also be applied to the well-known algorithms based on the Chinese remainder theorem (CRT) algorithm, which provides an explicit, noniterative formula for the UMR and m_0. Many approaches choose the measurement wavelengths to be integers in order to apply the CRT

[23,29–34]. The actual wavelength value used for the calculation is often truncated such that the wavelengths used in the calculation are represented by integers. Some variations and extensions of these methods utilize the fact that the ratio of measurement wavelengths can be represented as a rational number [40], as decimal continued fractions [41], or that the measurement wavelengths can be scaled by a known factor before truncation [36].

For all approaches, a well-known wavelength selection criterion is to choose the value of the individual wavelengths to be primes or coprimes in order to maximize the UMR [23,29,40]. The basic idea behind this method is that the individual wavelengths do not share common divisors and the point where all phase values of the wavelengths coincide is maximized. Hence, this criterion can be applied to both the CRT-based approaches as well as EF.

However, this effect does not take into account the system parameters δ and ψ, which vary strongly for various sets of primes and coprimes. In particular, if the smallest and largest measurement wavelengths are chosen to be very close together, the actual value of δ is low.

For instance, a three-wavelength integer interferometer with a phase noise of 1/300th of a fringe would violate the noise criterion of Eq. (33):

$$\delta \geq 6\sigma_R,$$

for any selection of wavelengths from the set of prime numbers within the range of a C.L. band source:

$$\{1531,1543,1549,1553,1559,1567,1571,1579\}.$$

The reason for the violation of the noise criterion of Eq. (33) lies in the limited maximum wavelength separation available given by the wavelengths $\lambda_0 = 1531\,\text{nm}$ and $\lambda_1 = 1579\,\text{nm}$, and the resulting scaling factor is larger than the permit able value given by the noise criterion. The situation can only be resolved if further laser sources are available spanning a larger wavelength range.

The other parameter ψ can depend on the relative magnitudes of ψ_0, δ, and γ (which in turn depends on α, sf, and m_0) and affects the minimum distance between potential fringe order solutions and the correct solution of the fringe order (see Fig. 4). The impact of the ψ parameter can be understood via an exemplary set of relative prime wavelengths $\{9, 10, 11\}$ proposed in [23] for fine resolution measurements. The corresponding value for α is 9/20, which means that the product of $(1-\alpha)z$ is exactly an integer at a distance of 20 beat wavelengths $z = 20$. However, the product $(1-\alpha)z$ becomes very close to an integer value for particular values of z, in this case at $z = 2, 4, 6, \ldots$, implying a considerably shorter UMR depending on the level of uncertainty in the phase.

In other MWI systems, the wavelengths have been selected so that $sf = 3.95$ and $\alpha = 0.5612$ where the phase noise is in the order of 1/240th of a fringe [34]. Because $\delta > 1/sf \gg 6(\sqrt{3}\sigma_\varepsilon)$, the error introduced by δ (which evaluates to $\delta = 1/3.61 \approx 38(\sqrt{3}\sigma_\varepsilon)$) is not statistically significant. Therefore, the reliability is only dependent on the parameter ψ, and reaches a minimum when the product $(1-\alpha)z$ becomes close to an integer. This is the case for $z = 7$ and $z = 9$, because α is close to the fractions $4/7 = 0.5714$ and $5/9 = 0.5556$. Again the practical UMR will be less than that expected.

B. Other Wavelength Selection Approaches

Another MWI has been presented by Decker *et al.* [6], having measurement wavelengths of $\lambda_0 = 543.515367\,\text{nm}$, $\lambda_1 = 611.9703403\,\text{nm}$, and $\lambda_2 = 632.9911815\,\text{nm}$; thus, $sf = 7.07$, $\alpha = 0.208653601$, and $\delta = 1/5.5441$ with a phase noise of 1/50th of a fringe. The value of δ is very close to the value required by the noise criterion of Eq. (33), and results in an overall reliability of 99.07%, if the measurement range is kept within the smallest beat wavelength, $\text{UMR} = [0, sf\lambda_0 - \lambda_0]$. The parameter ψ needs to be taken into account if the measurement range is to be extended. For instance, if the measurement range is kept within a $\text{UMR} = [0, 4sf\lambda_0 - \lambda_0]$, the corresponding value of ψ is equal to $\psi = 0.163$ (with the reliability of 99.89% for ψ) and the corresponding overall system reliability is equal to 98.98% (for δ and ψ). Another example is given for the case where the measurement range is equal to $\text{UMR} = [0, 23sf\lambda_0 - \lambda_0]$, with a minimum absolute value of ψ that is equal to $\psi = 2.7902e-002$ and the resulting overall reliability is reduced to 57.37%. Consequently, it has to be expected that when performing repetitive measurements over the range $162\lambda_0$ some of the calculated fringe orders are found to be incorrect, as it was observed in Decker *et al.* [6].

The introduction of a fourth wavelength, $\lambda_3 = 1152.59116\,\text{nm}$, helps to fulfill the noise criterion [Eq. (33)] as a result of a low value of δ. The system parameters of the four-wavelength interferometer are found to be $sf = 1.8924$, $\alpha_1 = 0.788$, $\alpha_2 = 0.733$, $\beta_1 = 0.791$, and $\delta = 1/1.999$. As a result of the large spacing between the smallest and largest measurement wavelength, δ is larger so that a reliability of 99.7% can be achieved considering this single parameter and the noise criterion of $\delta \geq 6\sqrt{4}\sigma_\varepsilon = 1/4.166$ is fulfilled. However, the behavior of the interferometer has changed for two reasons; first, the actual value of the scaling factor has changed, and second, the relative position of the center wavelengths with respect to the last wavelength has also changed. This set of wavelengths has a UMR of 28 fringe orders and a reliability of 99.6%. The UMR can be increased by compromising the reliability; for instance, the resulting performance is given as $\text{UMR} = 36\lambda_0$ (98.8% reliability), $\text{UMR} = 509\lambda_0$ (66.67% reliability), $\text{UMR} = 545\lambda_0$ (61.21% reliability), in accordance with the reported performance. It should be noted that, despite the fact that this set of four wavelengths is used to range over a remarkable distance of $545\lambda_0$,

the reliability can be increased with alternative wavelength selections. A reliability of 78.89% can be achieved with a phase noise of 1/50th of a fringe without compromising the UMR.

5. Summary

In this paper, a model of the method of EFs has been presented, which is applicable to a multiwavelength interferometer employing an arbitrary number of measurement wavelengths. Using this model, it is possible to define the quantities sf, δ, and ψ, which are directly related to the measurement wavelengths and describe the UMR and the system performance of the interferometer entirely. Generalized expressions have been developed to determine the minimum achievable UMR *a priori*, without carrying out simulations or experimental tests, even if the achievable UMR is found to be far beyond the largest beat wavelength. To the best of our knowledge, Eq. (27) is the first analytical expression for the UMR, which can be applied to arbitrary measurement wavelengths. Consequently, and unlike the case of integer interferometry, this procedure is not limited to sets of wavelengths where the values are truncated to integer values. It has also been shown that the NICF representation of $1/sf$ can be used directly to predict the minimum residual error value over any particular range in a computationally efficient manner. Furthermore, it is also possible to use sf, δ, and ψ to estimate quantitatively, for a given uncertainty in the phase measurements, the reliability of the interferometer without the use of simulations or experimental tests.

In experimental practice, the parameters sf, δ, and ψ can be set and tuned to any desired values which match the requirements of the given application. This requirement is also fulfilled with the optimum beat frequency approaches [20,24]. Nevertheless, the use of EF allows significantly more freedom in the choice of measurement wavelengths while delivering similar performance.

Appendix A

The work of Hurwitz [42] describes the NICF development from a real valued quantity x_0 as

$$x_0 = a_0 - \frac{1}{x_1}, \qquad x_1 = a_1 - \frac{1}{x_2}, \ldots, x_k = a_k - \frac{1}{x_{k+1}}, \ldots,$$

where $a_k = \mathrm{NINT}[x_k]$ and $\mathrm{NINT}(\cdot)$ is the nearest integer function. The fract function is useful in EF to define the residual error, $r_i(m_0)$, where $r_i(m_0) = \mathrm{fract}(m_i) = m_i - \mathrm{NINT}(m_i)$, hence the calculated fringe order which may be a real valued quantity is given by $m_i = \mathrm{NINT}(m_i) + \mathrm{fract}(m_i)$. Therefore a modified recurrence relation is defined for a real quantity x_0 as

$$x_0 = a_0 + \frac{1}{x_1}, \qquad x_1 = a_1 + \frac{1}{x_2}, \ldots, x_k = a_k + \frac{1}{x_{k+1}}, \ldots, \tag{A1}$$

where $a_k = \mathrm{NINT}[x_k]$ as before. The NICF of x_0 is given as

$$x_0 = a_0 + \frac{1}{x_1} = a_0 + \frac{1}{a_1 + \frac{1}{x_2}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{x_3}}}, \text{etc.}$$

A more compact form of the NICF of x_0 is given by the sequence

$$x_0 = (a_0, a_1, a_2, \ldots, a_k, x_{k+1}), \tag{A2}$$

where $k < K$, and $(K+1)$ is the number of elements of the sequence in Eq. (A2). It should be noted that K tends to infinity ($K \to \infty$) for the case of an irrational value of x_0. According to [42], the kth order approximation of x_0 is given by integer quantities s_k and q_k as

$$x_0 \approx \frac{s_k}{q_k} = (a_0, a_1, a_2, \ldots, a_k), \tag{A3}$$

which is the truncated sequence of Eq. (A2), and the quantities s_k and q_k are calculated by the recursion formulas

$$s_k = a_k s_{k-1} + s_{k-2}, \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}, \tag{A4}$$

with $s_{-2} = 0$, $s_{-1} = 1$, $q_{-1} = 0$, and $q_0 = 1$.

The recurrence relations in Eq. (A4) imply that the previously introduced quantities fulfil the identity

$$s_{k-1} q_k - q_{k-1} s_k = (-1)^k, \tag{A5}$$

$$x_0 = \frac{s_k x_{k+1} + s_{k-1}}{q_k x_{k+1} + q_{k-1}}. \tag{A6}$$

The convergence of the ratio "s_k/q_k" is equal to

$$x_0 - \frac{s_k}{q_k} = \left(\frac{1}{q_k}\right)^2 \frac{(-1)^k}{\left(x_{k+1} + \frac{q_{k-1}}{q_k}\right)}. \tag{A7}$$

For the special case of the recurrence relation defined by Eq. (A1), it is possible to calculate the residual error for the case of $k \geq 0$, which is interesting for MWI, as

$$\mathrm{fract}\left[\left(x_0 - \frac{s_k}{q_k}\right) q_k\right] = \frac{x_0}{p_{k+1}} (-1)^{k+1}, \tag{A8}$$

$$\mathrm{fract}\left[\left(\frac{1}{x_0} - \frac{q_k}{s_k}\right) s_k\right] = \frac{(-1)^k}{p_{k+1}}, \tag{A9}$$

where

$$p_k = \prod_{j=0}^{j=k} x_j. \tag{A10}$$

Appendix B: Application of Continued Fractions to the Evaluation of the fract(·) Function with Discrete Input Values

The absolute value of the total residual error of a two-wavelength interferometer is given as

$$R(m_0) = |\mathrm{fract}(m_0/sf)|, \tag{B1}$$

where m_0 is an integer and sf is a real number.

In experimental practice, it is important to estimate that range of possible values of m_0 for which the absolute value of the residual error falls below a certain threshold given by the measurement uncertainty of the interferometer. Usually, the residual error $R(m_0)$ is expected to be small when the product (m_0/sf) is close to an integer.

The critical values of m_0 at which the minimum residual error over the interval $m_0 = [1, m_{0\,\max}]$ can be calculated if the parameter $1/sf$ is developed with continued fractions. Hence, with the use of Eq. (A1) and by setting

$$x_0 = \frac{1}{sf}, \tag{B2}$$

low values of the residual error are found to be at $m_0 = |q_k|$.

It is also possible to describe any value of m_0 in the interval $m_0 = [|q_k|, |q_{k+1}|]$ as a linear combination of the quantities $|q_k|$ and $|q_{k+1}|$ as

$$m_0 = c_1|q_k| + c_2|q_{k+1}|, \tag{B3}$$

where c_1 and c_2 are integer constants.

The corresponding value of the residual error, using Eqs. (A8) and (B3), is given as

$$R(m_0 = c_1|q_k| + c_2|q_{k+1}|) = |\mathrm{fract}(c_1|q_k|x_0 + c_2|q_{k+1}|x_0)|,$$

so that

$$\begin{aligned} &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \\ &= \left|\mathrm{fract}\left(c_1 \mathrm{sign}[q_k] \frac{x_0}{p_{k+1}} (-1)^{k+1} \right.\right. \\ &\quad \left.\left. + c_2 \mathrm{sign}[q_{k+1}] \frac{x_0}{p_{k+2}} (-1)^k\right)\right|, \end{aligned} \tag{B4}$$

where $\mathrm{sign}(x)$ is the sign function defined as

$$\mathrm{sign}(x) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}. \tag{B5}$$

The three most critical values of $R(m_0)$ in the interval $m_0 = [|q_k|, |q_{k+1}|]$ are given as

$$\begin{aligned} &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) = x_0/p_{k+1} \quad (\text{for } m_0 = |q_k|), \\ &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \\ &\quad = x_0 \left| \frac{sign\,(q_{k+1})}{p_{k+2}} - \frac{sign\,(q_k)}{p_{k+1}} \right| \quad (\text{for } m_0 = |q_{k+1}| - |q_k|), \\ &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \\ &\quad = x_0/p_{k+2} \quad (\text{for } m_0 = |q_k|), \end{aligned} \tag{B6}$$

which can be described with the following lower boundary:

$$\begin{aligned} &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \geq x_0/p_{k+1} \quad (\text{for } m_0 = |q_k|), \\ &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \\ &\quad \geq \left| \frac{x_0}{p_{k+2}} - \frac{x_0}{p_{k+1}} \right| \quad (\text{for } m_0 = |q_{k+1}| - |q_k|), \\ &R(m_0 = c_1|q_k| + c_2|q_{k+1}|) \geq x_0/p_{k+2} \quad (\text{for } m_0 = |q_{k+1}|). \end{aligned} \tag{B7}$$

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References

1. A. A. Michelson and J. R. Benoit, "Détermination expérimentale de la valeur du mètre en longueurs d'ondes lumineuses," Trav. Et Mem. Bur. Int. Poids es Mes. **11**, 1–42 (1895).
2. J. Ye, "Absolute measurement of a long, arbitrary distance to less than an optical fringe," Opt. Lett. **29**, 1153–1155 (2004).
3. K.-N. Joo, Y. Kim, and S.-W. Kim, "Distance measurements by combined method based on a femtosecond pulse laser," Opt. Express **16**, 19799–19806 (2008).
4. Y. Salvadé, N. Schuhler, S. Lévêque, and S. Le Floch, "High-accuracy absolute distance measurement using frequency comb referenced multiwavelength source," Appl. Opt. **47**, 2715–2720 (2008).
5. O. P. Lay, S. Dubovitsky, R. D. Peters, J. P. Burger, S.-W. Ahn, W. H. Steier, H. R. Fetterman, and Y. Chang, "MSTAR: a submicrometer absolute metrology system," Opt. Lett. **28**, 890–892 (2003).
6. J. E. Decker, J. R. Miles, A. A. Madej, R. F. Siemsen, K. J. Siemsen, S. de Bonth, K. Bustraan, S. Temple, and J. R. Pekelsky, "Increasing the range of unambiguity in step-height measurement with multiple-wavelength interferometry-application to absolute long gauge block measurement," Appl. Opt. **42**, 5670–5678 (2003).
7. J. Jin, Y.-J. Kim, Y. Kim, S.-W. Kim, and C.-S. Kang, "Absolute length calibration of gauge blocks using optical comb of a femtosecond pulse laser," Opt. Express **14**, 5968–5974 (2006).
8. M. Shao, "Astrometry with SIM PlanetQuest," *Proceedings of the International Astronomical Union*, Vol. 3 (Cambridge University, 2007), pp. 231–237.
9. P. R. Lawson, O. P. Lay, K. J. Johnston, C. A. Beichman, S. C. Unwin, S. R. Martin, S. L. Hunyadi, O. Absil, R. L. Akeson, J. Bally, P. Bordé, D. Crisp, W. C. Danchi, D. Defrère, C. Eiroa, P. Falkowski, T. Henning, P. M. Hinz, J. M. Hollis, D. C. Hyland, L. Kaltenegger, L. Labadie, B. F. Lane, G. Laughlin, A. Léger, R. Liseau, D. Mawet, B. Mennesson, J. D. Monnier, A. C. Quillen, H. J. A. Röttgering, F. Selsis, E. Serabyn, D. J. Wilner, N. J. Woolf, W. A. Traub, R. O. Gappinger, A. Ksendzov, R. D. Peters, and D. P. Scharf, "Terrestrial Planet

Finder Interferometer (TPF-I) Whitepaper for the AAAC Exoplanet Task Force," http://planetquest1.jpl.nasa.gov/TPF-I/TPFIwhitepaper.pdf.
10. S. Lévéque, Y. Salvadé, R. Dandliker, and O. Scherler,"High-accuracy laser metrology enhances the VLTI," Laser Focus World **38**, 101–104 (2002).
11. K. Creath, "Phase measurement interferometry techniques," Progress in Optics **26**, 350–393 (1988).
12. M. Takeda, H. Ina, and S. Kobayashi, "Fourier-transform method of fringe-pattern analysis for computer-based topography and interferometry," J. Opt. Soc. Am. **72**, 156–160 (1982).
13. K. Falaggis, D. P. Towers, and C. E. Towers, "Theoretical analysis and optimisation of the method of excess fractions for long-range metrology," Proc. SPIE **7790**, 779009 (2010).
14. P. de Groot, "Design of error-compensating algorithms for sinusoidal phase shifting interferometry," Appl. Opt. **48**, 6788–6796 (2009).
15. K. E. Gillilland, H. D. Cook, K. D. Mielenz, and R. B. Stephens, "Use of a laser for length measurement by fringe counting," Metrologia **2**, 95–98 (1966).
16. C. R. Tilford, "Analytical procedure for determining lengths from fractional fringes," Appl. Opt. **16**, 1857–1860 (1977).
17. J. C. Wyant, "Testing aspherics using two-wavelength holography," Appl. Opt. **10**, 2113–2118 (1971).
18. Y.-Yen Cheng and J. C. Wyant, "Two-wavelength phase shifting interferometry," Appl. Opt. **23**, 4539–4543 (1984).
19. H. Zhao, W. Chen, and Y. Tan, "Phase-unwrapping algorithm for the measurement of three-dimensional object shapes," Appl. Opt. **33**, 4497–4500 (1994).
20. C. E. Towers, D. P. Towers, and J. D. C. Jones, "Optimum frequency selection in multifrequency interferometry," Opt. Lett. **28**, 887–889 (2003).
21. W. Nadeborn, P. Andra, and W. Osten, "A robust procedure for absolute phase measurement," Opt. Lasers Eng. **24**, 245–260 (1996).
22. J. M. Huntley and H. O. Saldner, "Error-reduction methods for shape measurement by temporal phase unwrapping," J. Opt. Soc. Am. A **14**, 3188–3196 (1997).
23. J. Burke, T. Bothe, W. Osten, and C. F. Hess, "Reverse engineering by fringe projection," Proc. SPIE **4778**, 312–324 (2002).
24. K. Falaggis, D. P. Towers, and C. E. Towers, "Multiwavelength interferometry: extended range metrology," Opt. Lett. **34**, 950–952 (2009).
25. P. J. de Groot, "Extending the unambiguous range of two-color interferometers," Appl. Opt. **33**, 5948–5953 (1994).
26. R. Benoît, "Application des phénomènes d'interférence à des déterminations métrologiques," Phys. Radium **7**, 57–68 (1898).
27. M. Born and E. Wolf, *Principles of Optics*, (Cambridge University, 2006), Chap. 7.6.
28. P. Hariharan, *Optical Interferometry*, (Academic, 1985).
29. V. I. Gushov and Y. N. Solodkin, "Automatic processing of fringe patterns in integer interferometers," Opt. Lasers Eng. **14**, 311–324 (1991).
30. J. Zhong and Y. Zhang, "Absolute phase-measurement technique based on number theory in multifrequency grating projection profilometry," Appl. Opt. **40**, 492–500 (2001).
31. J.-L. Li, H.-J. Su, and X.-Y. Su, "Two-frequency grating used in phase-measuring profilometry," Appl. Opt. **36**, 277–280 (1997).
32. Y. Surrel, "Two-step temporal phase unwrapping in profilometry" Proc. SPIE **3098**, 271–282 (1997).
33. Y. Hao, Y. Zhao, and D. Li, "Multifrequency grating projection profilometry based on the nonlinear excess fraction method," Appl. Opt. **38**, 4106–4110 (1999).
34. A. Pförtner and J. Schwider, "Red-green-blue interferometer for the metrology of discontinuous structures," Appl. Opt. **42**, 667–673 (2003).
35. C. E. Towers, D. P. Towers, and J. D. C. Jones, "Time efficient Chinese remainder theorem algorithm for full-field fringe phase analysis in multi-wavelength interferometry," Opt. Express **12**, 1136–1143.
36. C. E. Towers, D. P. Towers, and J. D. C. Jones, "Generalized multi-frequency selection for full-field interferometric shape measurement," Proc. SPIE **5502**, 406–409 (2004).
37. K. Falaggis, D. P. Towers, and C. E. Towers, "Optimum wavelength selection for the method of excess fractions," Proc. SPIE **7063**, 70630V (2008).
38. S. Hyun, Y.-J. Kim, Y. Kim, J. Jin, and S.-W. Kim, "Absolute length measurement with the frequency comb of a femtosecond laser," Meas. Sci. Technol. **20**, 095302 (2009).
39. J. M. Huntley, "Random phase measurement errors in digital speckle pattern interferometry," Opt. Lasers Eng. **26**, 131–150 (1997).
40. K. Houairi and F. Cassaing, "Two-wavelength interferometry: extended range and accurate optical path difference analytical estimator," J. Opt. Soc. Am. A **26**, 2503–2511 (2009).
41. M. Loefdahl and H. Eriksson, "An algorithm for resolving 2pi ambiguities in interferometric measurements by use of multiple wavelength," Opt. Eng. **40**, 984–990 (2001).
42. A. Hurwitz, "Über eine besondere Art der Kettenbruch-Entwicklung reeller Grössen," Acta Math. **12**, 367–405 (1889).